Abstract Algebra II
Revision

Below are some basic definitions one must know, theorems one must be able to prove, and questions one must be able to answer to pass the course. This list is not exhaustive.

Definitions

Definition. Solvability: Commutant, solvability, normal subgroup, simple
Groups: dihedral group, symmetric group (in terms of permutations and simplices), alternating group, cyclic group, quotient groups
Group structure: homomorphisms (iso-, auto-), kernel, image, parity (orientation),
Group actions: group actions (from axioms or permutations), orbits, stabilisers, p-groups,
Rings: Rings (from axioms or endomorphisms), ideals, quotient ring, field, principal ideal domain (PID), polynomial ring, extension ring,
Ring structure: homomorphisms (iso, auto-), kernel, image, maximal ideal, principal ideal, maximal ideal, prime and irreducible, Euclidean algorithm

Theorems

Theorem. *The following theorems should be known, without proofs:*

1. *Sylow's theorems;*
2. *A_5 is simple;*
3. *an ideal I is maximal in a ring R if and only if R/I is a field.*

Theorem. *The following theorems should be known, both the statements and proofs:*

1. *orbit-stabiliser theorem;*
2. *polynomial rings $F[x]$ over fields F are PID;*
3. *$\mathbb{Z}[\sqrt{-5}]$ is not PID;*
4. *$\mathbb{Z}[x]$ is not PID.*

Solvability

Question 1. Calculation of commutants of standard small groups, S_1–S_4, D_{2n} for even and odd n, C_n, A_1–A_4.

Question 2. Connection between commutativity and commutants.

Question 3. Solvability of basic small groups. Determine using any methods whether groups of specified order can be solvable or not.

Group actions

Question 4. Be familiar with basic group actions, trivial action of a group on itself, action of S_n on geometric objects, action by conjugation.

Question 5. Determining the size of groups using the orbit-stabiliser theorem.

Question 6. Know basic applications of Sylow's theorems:

- find normal subgroups of groups of certain order;
- study the group of symmetries of a cube (also reltated to the orbit-stabiliser theorem);
- determine solvability of certain groups.

Rings

Question 7. Determine whether a given ideal is maximal. Determine if a given ring is PID.

Question 8. Construct all maximal and principal ideals of a given ring.